# Introduction to Rigid Analytic Geometry

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What I want to convince you: analytification is very useful.

Example

Suppose $X/\mathbb{C}$ a smooth algebraic curve of genus $g$, then $X^{\mathrm{an}}$ is a complex manifold of dimension $1$, i.e. a Riemann surface, of genus $g$. In particular, when $X$ is an elliptic curve, $X^{\mathrm{an}}$ is a torus.

### Theorem (Uniformization theorem)

*The only simply connected Riemann surfaces are the Riemann sphere* $\mathbb{CP}^1$*, the complex plane* $\mathbb{C}$*, and the upper-half plane* $\mathbb{H}$*.*

Thus, we have the correspondence

| genus | $X$ | $X^{\mathrm{an}}$ |
|---|---|---|
| $0$ | conic section | $\mathbb{CP}^1$ |
| $1$ | elliptic curve | $\mathbb{C}/\Lambda$ |
| $\geq 2$ | modular curve | $\mathbb{H}/\Gamma$ |

where $\Lambda$ a lattice, and $\Gamma$ a congruence subgroup (congruent to $I$ mod some $N$) of $\mathrm{SL}_2(\mathbb{Z})$ acting by the Mobius transformation.

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### Theorem (Serre)

*For algebraic variety $X/\mathbb{C}$, we have equivalence of categories*

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(X^{\mathrm{an}})$$

*Moreover, there are isomorphisms*

$$H^q(X, \mathcal{F}) \cong H^q(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

*for each coherent sheaf $\mathcal{F}$ on $X$.*

## Definition

Recall a norm on a field $K$ is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that

1. $|x| = 0$ iff $x = 0$,
2. $|xy| = |x||y|$,
3. $|x + y| \leq |x| + |y|$,

for all $x, y \in K$, and moreover it is non-archimedean if

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## Theorem (Ostrowski)

*The nontrivial norms on $\mathbb{Q}$ are precisely the archimedean norm $|\cdot|$ and the nonarchimedean $p$-adic norms $|\cdot|_p$. Given by*

$$|n|_p = p^{-\mathrm{ord}_p(n)}$$

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However, the algebraic closure $\overline{\mathbb{Q}_p}$ is not complete, so one has to complete it again $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ to get a complete and algebraically closed non-archimedean field (Krasner's theorem).

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Is there a uniformization theorem or GAGA for this analogue?

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We will only talk about Tate's approach.

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We must eliminate these bad functions.

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Let $K$ a (complete) non-archimedean field, define the Tate algebra

$$K\langle X_1, \ldots, X_r \rangle = \left\{ \sum_{n \in \mathbb{N}^r} a_n \underline{X}^n : a_n \to 0 \text{ as } |n| \to \infty \right\}$$

An affnoid $K$-algebra is a quotient $K\langle X_1, \ldots, X_r \rangle / I$.

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This is a special case of a Grothendieck topology on a category.

Recall that a sheaf on a category with a Grothendieck topology $\mathcal{C}$ is a presheaf, i.e. contravariant functor $\mathcal{F} : \mathcal{C} \to \mathbf{Set}$, such that

$$\mathcal{F}(X) \to \prod_{\alpha \in A} \mathcal{F}(X_\alpha) \rightrightarrows \prod_{\alpha, \beta \in A} \mathcal{F}(X_\alpha \times_X X_\beta)$$

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Since the G-topology is a special case of Grothendieck topology, we can define sheaves on a G-topological space $X$, and a G-ringed spaces $(X, \mathcal{O}_X)$ as a G-topological space with sheaf of rings.

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A rigid analytic space defined by gluing affinoid domains, i.e. it is a G-ringed space locally isomorphic to affinoid domains.

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The good news is there is a rigid analytic analogue of GAGA.

And there is a so called Tate uniformization.

For an elliptic curve $E/\mathbb{C}$,

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The universal cover factors

$$\mathbb{C} \xrightarrow{e^{2\pi i z}} \mathbb{C}^\times \to \mathbb{C}^\times/q^{\mathbb{Z}} \cong E$$

where $q = e^{2\pi i \tau}$ satisfies $0 < |q| < 1$.

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is an elliptic curve. Moreover, by a theorem of Tate, an elliptic curve can be obtained this way iff it has bad reduction.

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and many more.

Thank you!