Adeles and Ideles

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1 Introduction

What is an *adele*? Besides a very famous English singer, an adele is also a mathematical object we associate to global fields used widely in algebraic number theory. Pioneered by Claude Chevalley and André Weil, adeles provide a natural language for many deep theorems in class field theory (so deep you could say they're *rolling in the deep*). We explain the basics of adeles in this write-up with some applications. We begin by recalling some basic facts from algebraic number theory. Fix K a number field (or more generally a global field, but we will focus on the number field case in this write-up).

Definition 1.1. An *absolute value* on a K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that

(i) $|x| = 0$ iff $x = 0$, for $x \in K$

(ii) $|xy| = |x||y|$, for $x, y \in K$

(iii) $|x + y| \leq |x| + |y|$, for $x, y \in K$

We say $|\cdot|$ is *non-archimedean* if it satisfies the ultrametric inequality

(iv) $|x + y| \leq \max\{|x|, |y|\}$, for $x, y \in K$

and *archimedean* otherwise. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are said to be *equivalent* if they induce the same topology on K, which happens iff $|\cdot|_1 = |\cdot|_2^\alpha$ for some $\alpha \in \mathbb{R}_{>0}$.

Theorem 1.2 (Ostrowski)**.** Any nontrivial absolute value on K is equivalent to either an archimedean absolute value induced by the usual absolute values on $\mathbb{R}$ or $\mathbb{C}$ via an embedding, or a non-archimedean $\mathfrak{p}$*-adic absolute value* $|\cdot|_\mathfrak{p}$ for a prime $\mathfrak{p} \subseteq \mathcal{O}_K$, where

$$|x|_\mathfrak{p} = c^{\mathrm{ord}_\mathfrak{p}(x)}$$

for a fixed $c \in (0, 1)$ for $x \in K^\times$ and $|0| = 0$. The constant c does not affect the equivalence class of the absolute value, but it is often set to $c = 1/N(\mathfrak{p})$ where $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ is the absolute norm of $\mathfrak{p}$, in which case we say it is *normalized*.

We call an equivalence class of absolute values on K a *place* of K. The archimedean places are called the *infinite places*, and the non-archimedean places are called *finite places*. Write K_v for the completion of K with respect to $|\cdot|_v$, and write $\mathcal{O}_v$ for its ring of integers.

Theorem 1.3. Let $\widehat{K}$ be the completion of K with respect to some nontrivial absolute value $|\cdot|$, and $\widehat{L}/\widehat{K}$ a finite extension of $\widehat{K}$, then $\widehat{L}$ is the completion of a finite extension L/K with respect to an absolute value that restricts to $|\cdot|$. In particular, when $|\cdot|$ is non-archimedean, the finite extensions of K_v are L_ω/K_v where L/K is a finite extension and $\omega \subseteq \mathcal{O}_L$ is a prime ideal such that $\omega | v$.

Adeles solves the technical problem of doing analysis on number fields over all the completions K_v simultaneously, and the way we achieve this is through a so-called restricted direct product.

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Definition 2.1. Define the *adele ring* of K as the restricted direct product

$$\mathbb{A}_K = \prod_\nu (K_\nu, \mathcal{O}_\nu)$$

that is, $\mathbb{A}_K$ is the subring of $\prod_\nu K_\nu$, where ν ranges over all places of K, that consists of all $(x_\nu)_\nu \in \prod_\nu K_\nu$ where $x_\nu \in \mathcal{O}_\nu$ for all but finitely many places ν.

The adele ring $\mathbb{A}_K$ is a locally compact topological ring, since every K_ν is locally compact. We would also like to consider the units $\mathbb{A}_K^\times$, but this is not a topological group since the inverse map $x \mapsto x^{-1}$ need not be continuous. Therefore, we must modify the topology on $\mathbb{A}_K^\times$.

Definition 2.2. Define the *idele group* of K as the group of units in its adele ring, i.e.

$$\mathbb{I}_K = \mathbb{A}_K^\times = \{(x_\nu)_\nu \in \mathbb{A}_K : x_\nu \in K_\nu^\times \text{ for all } \nu \text{ and } x_\nu \in \mathcal{O}_\nu^\times \text{ for all but finitely many } \nu\}$$

endowed with the weakest topology that makes it a topological group, that is, we embed

$$\mathbb{I}_K \to \mathbb{A}_K \times \mathbb{A}_K \quad x \mapsto (x, x^{-1})$$

and give $\mathbb{I}_K$ the subspace topology of its image in $\mathbb{A}_K \times \mathbb{A}_K$.

The injection $K \hookrightarrow \mathbb{A}_K$ restricts to inclusion $K^\times \hookrightarrow \mathbb{I}_K$, and $K^\times$ is a discrete locally compact subgroup of $\mathbb{I}_K$, thus we can define the *idele class group* $C(K) = \mathbb{I}_K / K^\times$. Let $\mathcal{I}_K$ be fractional ideals in $\mathcal{O}_K$, then there is a surjective morphism

$$\mathbb{I}_K \to \mathcal{I}_K \quad (a_\nu)_\nu \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(a)}$$

where $\nu_{\mathfrak{p}}(a) = \nu_{\mathfrak{p}}(a_\nu)$ where ν is the place corresponding to $|\cdot|_{\mathfrak{p}}$. The composition $K^\times \hookrightarrow \mathbb{I}_K \to \mathcal{I}_K$ has image $\mathcal{P}_K$ the principle fractional ideals. Thus the ideles class group surjects onto the ideal class group $\mathrm{Cl}(K) = \mathcal{I}_K / \mathcal{P}_K$.

Definition 2.3. Define the idele norm $\|\cdot\| : \mathbb{I}_K \to \mathbb{R}_{>0}^\times$ as

$$\|a\| = \prod_\nu |a|_\nu$$

where $|\cdot|_\nu$ is the normalized absolute value, and define the 1-idele group

$$\mathbb{I}_K^1 = \mathrm{Ker}(\|\cdot\|) = \{a \in \mathbb{I}_K : \|a\| = 1\}$$

as a subgroup of $\mathbb{I}_K$.

Lemma 2.4. Let L/K be a finite extension, ν a place of K, and ω a place of L where $\omega | \nu$, then we have

$$|x|_\omega = |N_{L_\omega / K_\nu}(x)|_\nu$$

Proof. Assume w.l.o.g. that $[L : K] > 1$. If ν is archimedean then $L_\omega = \mathbb{C}$ and $K_\nu = \mathbb{R}$, so $|N_{L_\omega/K_\nu}(x)|_\nu = |N_{\mathbb{C}/\mathbb{R}}(x)|_{\mathbb{R}} = x\overline{x} = |x|_{\mathbb{C}}^2 = |x|_\omega$. Assume ν is non-archimedean. Let π_ν and π_ω be the uniformizers of K_ν and L_ω respectively. Let f be the inertia degree of L_ω/K_ν. We have $x = \pi_\omega^{\omega(x)}$ w.l.o.g. Since

$$|N_{L_\omega/K_\nu}(\pi_\omega)|_\nu = |\pi_\nu^f|_\nu = |k_\nu|^{-f}$$

where k_ν is the residue field of K_ν, we have $|N_{L_\omega/K_\nu}(x)|_\nu = |k_\nu|^{-\omega(x)f}$. Thus

$$|x|_\omega = |k_\omega|^{-\omega(x)} = |k_\nu|^{-\omega(x)f} = |N_{L_\omega/K_\nu}(x)|_\nu$$

where k_ω is the residue field for L_ω. □

Lemma 2.5. Let L/K a finite extension then for ν a prime of K, we have

$$\prod_{\omega|\nu} |x|_\omega = |N_{L/K}(x)|_\nu$$

for $x \in L^\times$.

Proof. We note that $L \otimes_K K_\nu \cong \prod_{\omega|\nu}^n L_\omega$. To see this, let $L = K(\alpha) = K[x]/f(x)$ for primitive element $\alpha \in L$ with minimal polynomial f. Suppose f factors in $K_\nu[x]$ as irreducibles $f(x) = f_1(x)\cdots f_g(x)$. Then,

$$L \otimes_K K_\nu = K(\alpha) \otimes_K K_\nu \cong K_\nu[x]/f(x) \cong \prod_{i=1}^{g} K_\nu[x]/f_i(x) \cong \prod_{i=1}^{g} L_i$$

for finite extensions L_i/K_ν. By Theorem 1.3 and Hensel's lemma and Remark 8.3 in Milne [1], the L_i's are in fact L_ω for each $\omega|\nu$, therefore $L \otimes_K K_\nu \cong \prod_{\omega|\nu} L_\omega$. Therefore, we have $N_{L/K}(x) = \prod_{\omega|\nu} N_{L_\omega/K_\nu}(x)$, thus

$$|N_{L/K}(x)|_\nu = \prod_{\omega|\nu} |N_{L_\omega/K_\nu}(x)|_\nu = \prod_{\omega|\nu} |x|_\omega$$

by Lemma 2.4. □

Theorem 2.6 (Artin's Product Formula)**.** We have

$$\|x\| = \prod_\nu |x|_\nu = 1$$

for all $x \in K^\times$. In other words, there is canonical inclusion $K^\times \hookrightarrow \mathbb{I}_K^1$.

Proof. We first prove this for $K = \mathbb{Q}$. Let $x = a/b \in \mathbb{Q}$ for $a, b \in \mathbb{Z}$. Since $|x|_p = 1$ unless p divides one of a, b, the product is finite. The map $\|\cdot\| : \mathbb{Q}^\times \to \mathbb{R}_{>0}^\times$ is a homomorphism, so it suffice to check that $\|-1\| = 1$ and $\|p\| = 1$ for all prime p. The first is obvious, and for the second, note that $|p|_p = 1/p$, $|p|_\infty = p$, and $|p|_q = 1$ when $q \neq p$ is prime. Next, by Lemma 2.5, to prove the case for a number field $L/\mathbb{Q}$, it suffice to show that $\prod_\nu |N_{L/\mathbb{Q}}(x)|_\nu = 1$ for $x \in L^\times$, which we have already shown. □

Lemma 2.7. The 1-idele group $\mathbb{I}_K^1$ inherits the same topology regardless whether we view it in $\mathbb{I}_K$ (where it is a closed subgroup) or $\mathbb{A}_K$ (where it is closed).

Proof. First we show $\mathbb{I}_K^1$ is closed in $\mathbb{A}_K$. To this end, we seek an open neighborhood N of each $x \in \mathbb{A}_K \setminus \mathbb{I}_K^1$ disjoint from $\mathbb{I}_K^1$. Choose $x \in (x_\nu) \in \mathbb{A}_K \setminus \mathbb{I}_K^1$. Let S be a finite set of places that includes all archimedean places such that $x_\nu \in \mathcal{O}_\nu$ for all $\nu \notin S$.

Assume $x_{\nu_0} = 0$ for some ν_0. Consider $N = \prod_\nu N_\nu$ where $N_\nu = \mathcal{O}_\nu$ for $\nu \notin S \cup \{\nu_0\}$, N_ν is a compact neighborhood of x_ν in K_ν for $\nu \in S$ with $\nu \neq \nu_0$, and N_{ν_0} a very small neighborhood of $0 \in K_{\nu_0}$. By very small, we mean small enough so that any idele in N has idelic norm very close to 0 and in particular smaller than 1 (note that we can do this because any idele in N has idelic norm bounded by product of local norms along places in S and ν_0), so that N is disjoint from $\mathbb{I}_K^1$.

Next, assume $x_\nu \in K_\nu^\times$ for all ν, since $|x_\nu|_\nu \leq 1$ for all but finitely many ν, the infinite product $\|x\| = \prod_\nu |x|_\nu$ has partial products eventually forming a monotonically decreasing sequence, and hence this product makes sense as a nonnegative real number. If $\|x\| < 1$, then let S be a finite set of places containing all archimedean places and such that $x_\nu \in \mathcal{O}_\nu$ for all $\nu \notin S$, and $\prod_{\nu \in S} |x_\nu|_\nu < 1$. Now, we can apply the same argument as the preceeding paragraph: we take $N_\nu = \mathcal{O}_\nu$ for $\nu \notin S$ and choose N_ν a very small neighborhood for x_ν for $\nu \in S$. If instead the product $P = \|x\| > 1$. By the convergence of this product, there is a finite set of places S including all archimedean places, such that $x_\nu \in \mathcal{O}_\nu$ for all $v \notin S$, and $\frac{1}{2P} > \frac{1}{q_\nu}$ for $\nu \notin S$ (so that $|\xi_\nu|_\nu < 1$ implies $|\xi_\nu|_\nu < \frac{1}{2P}$), and $1 < \prod_{\nu \in S} |x_\nu|_\nu < 2P$.

Hence by choosing $N_\nu = \mathcal{O}_\nu$ for $\nu \notin S$, and choosing N_ν very small for $\nu \in S$, we have if $\xi \in N$, and $|\xi_\nu|_\nu < 1$ for some $\nu \notin S$, then

$$\|\xi\| = \prod_{\nu \in S} |\xi_\nu|_\nu \cdot \prod_{\nu \notin S} |\xi_\nu|_\nu < \frac{2P}{2P} = 1$$

and if $|\xi_\nu|_\nu \geq 1$ for all $\nu \notin S$ then $\|\xi\| > 1$, so it is also disjoint from $\mathbb{I}_K^1$.

Next, we show $\mathbb{I}_K^1 \hookrightarrow \mathbb{A}_K$ is a homeomorphism onto its closed image. It suffice to show that every $N \subseteq \mathbb{I}_K^1$ around a point x contains the intersection of $\mathbb{I}_K^1$ with a neighborhood of x in $\mathbb{A}_K$. We may assume that $x = 1$ since multiplication by $1/x$ is a an automorphism that carries $\mathbb{I}_K^1$ onto itself. By the description of the neighborhood basis of $1 \in I_K$, we can shrink N as $N = \prod_\nu N_\nu \cap \mathbb{I}_K^1$ with each N_ν a small disc centred at 1 in $K_\nu^\times$ for all ν in a finite set of places S containing all archimedean places, and $N_\nu = \mathcal{O}_\nu^\times$ for all $\nu \notin S$. By shrinking N_ν, we can ensure that $\prod_{\nu \in S} |\xi_\nu|_\nu < 2$ for all $\xi \in \prod_\nu N_\nu \subseteq \mathbb{I}_K$, yet for $\xi \in \prod_\nu N_\nu$ we have $\|\xi\| = \prod_{\nu \in S} |\xi_\nu|_\nu$ since $N_\nu = \mathcal{O}_\nu^\times$ for $\nu \notin S$. Let W be the neighborhood $\prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} \mathcal{O}_\nu$ of 1 in $\mathbb{A}_K$, then any $\xi \in W \cap \mathbb{I}_K^1$ satisfies

$$1 = \|\xi\| = \prod_{\nu \in S} |\xi_\nu|_\nu \cdot \prod_{\nu \notin S} |\xi_\nu|_\nu < 2 \prod_{v \notin S} |\xi_\nu|_\nu$$

Since $|\xi_\nu|_\nu \leq 1$ for $\nu \notin S$, $1 < 2|\xi_{\nu_0}|_{\nu_0}$ for $\nu_0 \notin S$, so $1 \geq |\xi_{\nu_0}| > 1/2 \geq 1/q_{\nu_0}$, so $\xi_{\nu_0} \in \mathcal{O}_{\nu_0}^\times = N_{\nu_0}$ so $W \cap \mathbb{I}_K^1 \subseteq \prod_\nu N_\nu \cap \mathbb{I}_K^1 = N$. □

Theorem 2.8 (Adelic Minkowski's theorem)**.** For $\xi \in \mathbb{I}_K$, define the closed subset

$$X_\xi = \{(x_v) \in \mathbb{A}_K : |x_v|_v \leq |\xi_v|_v\} \subseteq \mathbb{A}_K$$

then there eixsts $C > 0$ such that if $\|\xi\| > C$ then $X_\xi \cap K$ has a nonzero element.

Lemma 2.9. The quotient $\mathbb{I}_K^1 / K^\times$ is compact.

Proof. By Lemma 2.7, the topology on $\mathbb{I}_K^1$ is induced by $\mathbb{A}_K$. Thus for any compact $W \subseteq \mathbb{A}_K$, we have $W \cap \mathbb{I}_K^1$ is compact. It suffice to find compact W for which the projection $W \cap \mathbb{I}_K^1 \to \mathbb{I}_K^1 / K^\times$ is surjective. Choose $C > 0$ as in Theorem 2.8 and $\xi \in \mathbb{I}_K$ such that $\|\xi\| > C$, define

$$W = \{x \in \mathbb{A}_K : |x_v|_v \leq |\xi_v|_v\}$$

For any $\theta \in \mathbb{I}_K^1$, the idele $\theta^{-1}\xi$ has idelic norm $\|\xi\| > C$. By Theorem 2.8, there exists nonzero $a \in K$ such that $|a|_v \leq |\theta^{-1}\xi_v|_v$ for all v, and hence $a\theta \in W$. Since $a \in K^\times \subseteq \mathbb{I}_K^1$, so $a\theta \in W \cap \mathbb{I}_K^1$ is a representative of the class of θ in $\mathbb{I}_K^1 / K^\times$. □

3 Dirichlet Unit Theorem and Class Number

The machinery of adeles has application in proving several finiteness results. Let S be a set of places that includes all archimedean places, then the S-adele ring $\mathbb{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$ is an open subring of $\mathbb{A}_K$. The S-integers $\mathcal{O}_{K,S}$ is the set of $a \in K$ such that a is v-integral for $v \notin S$.

Theorem 3.1. The following are equivalent

1. $\mathbb{I}_K^1 / K^\times$ is compact
2. $\mathrm{Cl}(\mathcal{O}_{K,S})$ is finite and $\mathcal{O}_{K,S}^\times$ has rank $|S| - 1$.

Proof. See Conrad [2]. □

References

[1] J. S. Milne, *Algebraic Number Theory*.

[2] B. Conrad, *Algebraic Number Theory*.

[3] J. Neukirch, *Algebraic Number Theory*.

[4] A. Sutherland, *18.785 - Number Theory I*.