Galois Category & Étale Fundamental Group

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Motivation

We begin by some motivations from algebraic topology.

Recall the fundamental group of a topological space X at $x \in X$

$$\pi_1(x, X) = \{\textit{cont. } \gamma : [0,1] \to X \textit{ where } \gamma(0) = \gamma(1) = x\}/\sim$$

where $\gamma \sim \tau$ iff their homotopic, with concatonation as group operation. Recall a covering space $p : E \to X$ is a fiber bundle with discrete fibers.

It is shown that the fundamental group is isomorphic to the group of deck transformations of the universal covering space.

In fact, there is a well known stronger result.

Theorem 1.1 (fundamental theorem of covering spaces)

Suppose X is path-connected, locally path-connected, and semi-locally simply-connected topological space, then there is an equivalence between

i. *the category of covering spaces of X, and*
ii. *the category of $\pi_1(x, X)$-sets, and*
iii. *the category of subgroups of $\pi_1(x, X)$ up to conjugacy.*

This is strikingly reminiscent of Galois theory!

Recall that given a Galois extension E/F, the Galois group

$$\mathrm{Gal}(E/F) = \{\textit{isom. } \alpha : E \to E : \forall x \in F, \alpha(x) = x\}$$

with composition as group operation.

The fundamental theorem of Galois theory reveals a correspondence between Galois extensions and subgroups of the Galois group, which resembles the fundamental theorem of covering spaces.

Theorem 1.2 (fundamental theorem of Galois theory)

Suppose k is a field with $K = \overline{k}$, then there is an equivalence between

i. *the category of Galois extensions of k, and*
ii. *the category* $\mathrm{Gal}(K/k)$*-sets, and*
iii. *the category of subgroups of* $\mathrm{Gal}(K/k)$.

This is a categorical reformulation of the well known version.

We observe that X is analogous to Y, where

X	Y
Galois extensions	covering spaces
algebraic closure	universal covering space
Galois group	fundamental group
k-automorphisms	deck transformations
fun'l theorem of Galois theory	fun'l theorem of covering spaces

There is only a superficial difference between the two fundamental theorems: one isomorphism is covariant and the other is contravariant.

We will discuss *étale fundamental groups* of schemes, which is a generalization of both the Galois groups and the topological fundamental groups. However, we must first introduce some necessary backgrounds.

Galois category

Suppose $\mathcal{C}$ is a category and $\mathcal{F} : \mathcal{C} \to \textbf{Set}$ a functor, then there is a canonical injective function

$$\mathrm{Aut}(\mathcal{F}) \longrightarrow \prod_{X \in \mathcal{C}} \mathrm{Aut}(\mathcal{F}(X)) \qquad \phi \mapsto [X \mapsto \phi_X]$$

For each set E, we give $\mathrm{Aut}(E)$ the compact open topology (i.e. the coarsest topology such that $\mathrm{Aut}(E) \times E \to E$ by $(f, e) \mapsto f(e)$ is cont.), which is discrete when E is finite. Next, we endow $\mathrm{Aut}(\mathcal{F})$ the topology induced from the canonical map above.

This map identifies $\mathrm{Aut}(\mathcal{F})$ with a closed subgroup of $\prod_{X \in \mathcal{C}} \mathrm{Aut}(\mathcal{F}(X))$, and in particular if $\mathcal{F}(X)$ is finite for all X, then $\mathrm{Aut}(\mathcal{F})$ is profinite.

Definition 2.3 (profinite completion)

The *profinite completion* of a topological group G is

$$G^{\wedge} = \varprojlim_{\substack{\textit{open } U \trianglelefteq G \\ \textit{of finite index}}} G/U$$

has the following universal property: every morphism $G \to H$ where H is profinite factors uniquely as $G \to G^{\wedge} \to H$.

Lemma 2.4

Let G be a topological group, and $\mathcal{F} : \mathrm{Rep}_{\mathbf{FSet}}(G) \to \mathbf{Set}$ the forgetful functor where $\mathrm{Rep}_{\mathbf{FSet}}(G)$ is the category of finite G-sets, then

$$G^{\wedge} \cong \mathrm{Aut}(\mathcal{F})$$

as topological groups.

Sketch of Proof.

Take the canonical morphism $G \to \mathrm{Aut}(\mathcal{F})$ of topological groups, there is an induced $G^{\wedge} \to \mathrm{Aut}(\mathcal{F})$ via universal property, which is shown to be injective, and is a homeomorphism if shown its image is dense by Lemma 5.17.8 [Tag 08YE] of Stacks Project. This is shown by showing that for every $\gamma \in \mathrm{Aut}(\mathcal{F})$ and finite G-set X, there is $g \in G$ s.t. g and γ induce the same action on $\mathcal{F}(X)$. □

Lemma 2.5

Let G be a topological group, and $\mathcal{F} : \mathrm{Rep}_{\mathbf{FSet}}(G) \to \mathbf{Set}$ be an exact functor where $\mathcal{F}(X)$ is finite for all X, then $\mathcal{F}$ is isomorphic to the forgetful functor $\mathrm{Rep}_{\mathbf{FSet}}(G) \to \mathbf{Set}$.

Sketch of Proof.

After showing the inverse limit over $\mathcal{F}(G/U)$ for all open $U \trianglelefteq G$ of finite index is nonempty, pick an element γ. Identify the forgetful functor with

$$X \mapsto \varinjlim_{\substack{\text{open } U \trianglelefteq G \\ \text{of finite index}}} \mathrm{Mor}(G/U, X)$$

where $f : G/U \to X$ corresponds to $f(eU)$. Thus γ induces an isomorphism t from the forgetful functor to $\mathcal{F}$ as follows: for $x \in X$ choose U and $f : G/U \to X$ sending eU to x, let $t_X(x) = \mathcal{F}(f)(\gamma_U)$. □

Example 2.6

Let $\mathcal{C}$ be a category and $\mathcal{F} : \mathcal{C} \to \mathbf{Set}$ a functor with $\mathcal{F}(X)$ finite for all $X \in \mathcal{C}$. Previously we defined the profinite topological group $\mathrm{Aut}(\mathcal{F})$, thus there is a canonical functor

$$\mathcal{C} \to \mathrm{Rep}_{\mathbf{FSet}}(\mathrm{Aut}(\mathcal{F})) \qquad X \mapsto \mathcal{F}(X)$$

endowed with the induced action of $\mathrm{Aut}(\mathcal{F})$.

The reason we need Galois category is that we want to single out the categories $\mathcal{C}$ and functors $\mathfrak{F}$ as in Example 2.6 where the canonical functor defined is an equivalence of categories.

Definition 2.7 (Galois category)

A *Galois category* is a category $\mathcal{C}$ with a functor $\mathcal{F} : \mathcal{C} \to \mathbf{Set}$ s.t.

i. $\mathcal{F}(X)$ is finite for all X, and
ii. $\mathcal{C}$ has finite limits and colimits, and
iii. $\mathcal{F}$ is exact and conservative, and
iv. every object of $\mathcal{C}$ is a finite coproduct of connected components.

Lemma 2.8 (properties of Galois categories)

Suppose $\langle \mathcal{C}, \mathcal{F} \rangle$ *is a Galois category, then*

i. $\mathcal{F}$ *is faithful*
ii. $\mathcal{F}$ *preserves mono/epimorphisms and initial/final objects*
iii. *if* X, Y *are connected then* $X \to Y$ *is epic,*
iv. *if* X *is connected, suppose* $a, b : X \to Y$ *are two morphisms, then* $a = b$ *if* $\mathcal{F}(a)$ *and* $\mathcal{F}(b)$ *agree on one element of* $\mathcal{F}(X)$.

Proof.

Lemma 58.3.7 [Tag 0BN0] of Stacks Project. □

Given a Galois category $\langle \mathcal{C}, \mathcal{F} \rangle$, from Lemma 2.8, if X is connected,

$$|\operatorname{Aut}(X)| \leq |\mathcal{F}(X)|$$

We say that X is *Galois* if equality holds. It is straightforward to see that X is Galois iff X is connected and $\operatorname{Aut}(X)$ acts transitively on $\mathcal{F}(X)$.

The following is a key lemma.

Lemma 2.9

Let $\langle \mathcal{C}, \mathcal{F} \rangle$ be a Galois category with connected object X, then

i. *there exists Galois object Y and morphism $Y \to X$,*
ii. *the action of $\mathrm{Aut}(\mathcal{F})$ on $\mathcal{F}(X)$ (as per Example 2.6) is transitive,*
iii. *the canonical functor (as per Example 2.6) is an equivalence,*
iv. *suppose $\langle \mathcal{D}, \mathcal{G} \rangle$ is a Galois category, $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ is an exact functor, then there is isomorphism $t : \mathcal{G} \circ \mathcal{H} \to \mathcal{F}$.*

Proof.

The proof is long and technical, see Lemma 58.3.8 [Tag 0BN2] to Lemma 58.3.11 [Tag 0BN5] of Stacks Project. □

Finite étale morphism

Recall from class the definition of étale morphisms.

Definition 3.10 (étale morphisms)

A morphism of schemes is étale iff it is

i. smooth and unramified, or
ii. smooth and locally quasi-finite, or
iii. flat and unramified, or
iv. formally étale and locally of finite presentation,

where all these conditions are equivalent.

An étale morphism is analogous to a covering map in topology.

Recall an important example from class.

Example 3.11

Suppose X is a scheme over k, then its structure morphism is étale iff

$$X \cong \coprod_i \operatorname{Spec}(K_i)$$

where K_i are finite separable extensions of k.

Proof.

Lemma 29.36.7(1) [Tag 02GL] of Stacks Project. □

Lemma 3.12 (properties of étale morphisms)

i. *composition of étale morphisms is étale,*
ii. *base change of étale morphisms is étale,*
iii. *étale morphisms are local on the source and the base,*
iv. *if $X \to Y \to Z$ and $Y \to Z$ are étale, then so is $X \to Y$.*

Proof.

Proved in class. □

Suppose X is a scheme, we use $\mathbf{FÉt}_X$ to denote the full subcategory of $\mathbf{Sch}/X$ where the objects are finite and étale. An object $Y \to X$ of $\mathbf{FÉt}_X$ is a *Galois cover* if Y is a Galois object.

Lemma 3.13

The category $\mathbf{FÉt}_X$

i. has finite limits and colimits,

ii. for any $Y \to X$*, the base change functor* $\mathbf{FÉt}_X \to \mathbf{FÉt}_Y$ *is exact.*

Sketch of Proof.

First, $\mathbf{FÉt}_X$ *has final object and admits fibred products, hence admits finite limits, and base change is left exact as they commute with them. Next,* $\mathbf{FÉt}_X$ *admits finite coproducts (disjoint unions). It suffice to show that* $\mathbf{FÉt}_X$ *admits coequalizers, so that* $\mathbf{FÉt}_X$ *admits colimits and the base change functor is right exact. This is done by a direct construction.* □

Let X be a connected scheme with $\overline{x}$ a geometric point, then the functor

$$\mathcal{F}_{\overline{x}} : \mathbf{F\acute{E}t}_X \to \mathbf{Set} \qquad Y \mapsto |Y_{\overline{x}}|$$

defines a Galois category. This functor is called the fibre functor.

We verify $\langle \mathbf{F\acute{E}t}_X, \mathcal{F}_{\overline{x}} \rangle$ satisfies Definition 2.7. First $\mathbf{F\acute{E}t}_{\overline{x}} \cong \mathbf{FSet}$ by Example 3.11. Thus $\mathcal{F}_{\overline{x}}$ is a base change functor for $\overline{x} \to X$, so by Lemma 3.13 $\mathcal{F}_{\overline{x}}$ is exact (and $\mathbf{F\acute{E}t}_X$ admits finite limits and colimits). We can verify that each object is connected iff they are connected as a scheme, so they are finite coproduct of connected components by Lemma 5.7.7 [Tag 07VB] of Stacks Project. It is straightforward to verify that $\mathcal{F}_{\overline{x}}$ is conservative.

Étale fundamental group

Definition 4.14 (étale fundamental group)

Suppose that X is a connected scheme and $\overline{x}$ is a geometric point, then the *(étale) fundamental group* of X at $\overline{x}$ is defined as

$$\pi_1(\overline{x}, X) = \mathrm{Aut}(\mathcal{F}_{\overline{x}})$$

endowed with the canonical profinite topology (as per Section 2), where $\mathcal{F}_{\overline{x}}$ is the fiber functor (as per Section 3).

Theorem 4.15 (main theorem)

Suppose X is a connected scheme with geometric point $\overline{x}$.

i. *the functor $\mathcal{F}_{\overline{x}}$ induces a canonical equivalence (as per Example 2.6)*

$$\mathbf{F\acute{E}t}_X \cong \mathrm{Rep}_{\mathbf{FSet}}(\pi_1(\overline{x}, X))$$

ii. *given connected scheme Y and morphism $f : X \to Y$ and let $\overline{y} = f \circ \overline{x}$, then there is a canonical morphism $f_* : \pi_1(\overline{x}, X) \to \pi_1(\overline{y}, Y)$ s.t.*

$$\begin{array}{ccc} \mathbf{F\acute{E}t}_Y & \xrightarrow{\text{base change}} & \mathbf{F\acute{E}t}_X \\ \downarrow{\scriptstyle \mathcal{F}_{\overline{y}}} & & \downarrow{\scriptstyle \mathcal{F}_{\overline{x}}} \\ \mathrm{Rep}_{\mathbf{FSet}}(\pi_1(\overline{y}, Y)) & \xrightarrow[f_*]{} & \mathrm{Rep}_{\mathbf{FSet}}(\pi_1(\overline{x}, X)) \end{array}$$

commutes.

Theorem 4.15 explains the correspondances we introduced in Section 1.

Thank you for listening!

References

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